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Rotating black holes can have short bristles

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ABSTRACT

The elegant ‘no short hair’ theorem states that, if a spherically-symmetric static black hole has hair, then this hair must extend beyond $3/2$ the horizon radius. In the present paper we provide evidence for the failure of this theorem beyond the regime of spherically-symmetric static black holes. In particular, we show that rotating black holes can support extremely short-range stationary scalar configurations (linearized scalar ‘clouds’) in their exterior regions. To that end, we solve analytically the Klein–Gordon–Kerr–Newman wave equation for a linearized massive scalar field in the regime of large scalar masses.

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1. Introduction

Within the framework of classical general relativity, the black-hole horizon acts as a one-way membrane which irreversibly absorbs matter fields and radiation. This remarkable property of the black-hole horizon suggests, in particular, that static matter configurations cannot be supported in the spacetime region outside the black-hole horizon. This expectation is nicely summarized in Wheeler’s famous dictum “a black hole has no hair” [1,2], which suggests that the spacetime geometries of all asymptotically flat stationary black holes are uniquely described by the three-parameter family [3] of the Kerr–Newman electrovacuum solution [4–6].

The ‘no-hair’ conjecture [1,2] has attracted much attention over the years from both physicists and mathematicians. Early investigations of the conjecture have ruled out the existence of static hairy black-hole configurations made of scalar fields [7], spinor fields [8], and massive vector fields [9]. However, the early 90s have witnessed the discovery of a variety of regular [10] hairy black-hole configurations, the first of which were the ‘colored’ black holes which are solutions of the coupled Einstein–Yang–Mills equations [11]. It has soon been realized that many non-linear matter fields [12], when coupled to the Einstein field equations, can lead to the formation of hairy black-hole configurations [13–23].

The validity of the original no-hair conjecture [1,2] has become highly doubtful since the discovery of these non-linear [11,13–23] hairy black-hole configurations [24]. The current situation naturally

gives rise to the following question: Is it possible to formulate a more modest (and robust) alternative to the original no hair conjecture?

A very intriguing attempt to reveal the generic characteristics of hairy black-hole configurations was made in [25]: A ‘no short hair’ theorem was proved, according to which static spherically-symmetric black holes cannot support short hair. In particular, it was shown in [25] that, in all Einstein–matter theories in which static hairy black-hole configurations have been discovered, the effective length of the outside hair is bounded from below by [26]

$$r_{\text{hair}} > \frac{3}{2} r_H, \quad (1)$$

where r_H is the horizon-radius of the black hole. This ‘no short hair’ theorem was suggested [25] as an alternative to the original [1,2] ‘no hair’ conjecture.

It is worth emphasizing that the formal proof of the lower bound (1) provided in [25] is restricted to the static sector of spherically-symmetric black holes. Nevertheless, it was conjectured [25] that the ‘no short hair’ bound (1) can be generalized in the form

$$r_{\text{hair}} > \frac{3}{2} \sqrt{\frac{A_H}{4\pi}}; \quad A_H \equiv \text{horizon area} \quad (2)$$

to include the cases of non-spherically-symmetric stationary hairy black-hole configurations.

The main goal of the present paper is to test the validity of the ‘no short hair’ conjecture beyond the regime of spherically symmetric static black holes. In particular, we shall explore here the physical properties of non-spherically-symmetric rotating black holes coupled to linearized stationary (rather than static) scalar

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matter configurations. (It should be emphasized that the scalar fields we consider have a time dependence of the form $e^{-i\omega t}$ [see Eq. (10) below]. However, physical quantities, like the energy-momentum tensor itself, are time-independent.)

2. Composed black-hole–scalar-field configurations

While early no hair theorems have shown that asymptotically flat black holes cannot support regular static scalar configurations in their exterior regions [7], they have not ruled out the existence of non-static composed black-hole–scalar-field configurations. In fact, it has recently [27] been demonstrated that rotating black holes can support linearized stationary scalar configurations (scalar ‘clouds’ [28,29]) in their exterior regions. Since non-linear (self-interaction) effects tend to stabilize the outside hair [25,30], we conjectured in [27] the existence of rotating black hole solutions endowed with genuine non-static scalar hair. These non-static hairy black-hole–scalar-field configurations are the non-linear counterparts of the linear scalar clouds studied analytically in [27]. In a very interesting letter, Herdeiro and Radu [31] have recently solved numerically the non-linear coupled Einstein–scalar equations, and confirmed the existence of these non-static hairy black-hole configurations.

The composed black-hole–scalar-field configurations [32] explored in [27,31] are intimately related to the intriguing phenomenon of superradiant scattering of bosonic fields in rotating black-hole spacetimes [33–36]. In particular, the linearized stationary scalar configurations studied in [27,31] are characterized by orbital frequencies which are integer multiples of the central black-hole angular frequency [37]:

$$\omega_{\text{field}} = m\Omega_H \quad \text{with } m = 1, 2, 3, \dots \quad (3)$$

It is well-established [33–36] that the energy flux of the field into the central spinning black hole vanishes for bosonic modes which satisfy the relation (3). In this case, the bosonic field is not swallowed by the central black hole. This suggests that stationary bosonic configurations which are in resonance with the central spinning black hole (that is, bosonic fields with orbital frequencies $\omega_{\text{field}} = m\Omega_H$) may survive in the spacetime region exterior to the black-hole horizon.

In order to have genuine stationary (non-decaying) field configurations around the central black hole, one should also prevent the field from escaping to infinity. A natural confinement mechanism is provided by the gravitational attraction between the massive field and the central black hole. In particular, for a scalar field of mass μ , low frequency field modes in the regime [38]

$$\omega^2 < \mu^2 \quad (4)$$

are confined to the vicinity of the central black hole.

As discussed above, the main goal of the present paper is to test the validity of the ‘no short hair’ conjecture (1) [25] beyond the regime of spherically-symmetric static black holes. To that end, we shall analyze the physical properties of the non-static (rotating) black-hole–scalar-field configurations [27,31] in the eikonal regime

$$M\mu \gg 1, \quad (5)$$

where M is the mass of the central spinning black hole.

3. Description of the system

The physical system we consider consists of a massive scalar field ψ linearly coupled [39] to an extremal Kerr–Newman black hole of mass M , angular-momentum per unit mass a , and electric

charge Q . In Boyer–Lindquist coordinates (t, r, θ, ϕ) the spacetime metric is given by [4–6]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2}[adt - (r^2 + a^2)d\phi]^2 \quad (6)$$

where $\Delta \equiv r^2 - 2Mr + a^2 + Q^2$ and $\rho \equiv r^2 + a^2 \cos^2 \theta$. The extremality condition implies that the degenerate horizon of the black hole is located at

$$r_H = M = \sqrt{a^2 + Q^2}. \quad (7)$$

The angular velocity of the black hole is given by [4–6]

$$\Omega_H = \frac{a}{M^2 + a^2}. \quad (8)$$

The dynamics of the linearized massive scalar field ψ in the Kerr–Newman black-hole spacetime is governed by the Klein–Gordon (Teukolsky) wave equation

$$(\nabla^\nu \nabla_\nu - \mu^2)\psi = 0. \quad (9)$$

It proves useful to use the ansatz [40]

$$\psi(t, r, \theta, \phi) = \int \sum_{l,m} e^{im\phi} S_{lm}(\theta; s\epsilon) R_{lm}(r; s, \mu, \omega) e^{-i\omega t} d\omega \quad (10)$$

for the scalar wave field in (9), where

$$s \equiv \frac{a}{M} \quad (11)$$

is the dimensionless angular-momentum (spin) of the black hole, and

$$\epsilon \equiv M\sqrt{\mu^2 - \omega^2}. \quad (12)$$

The angular equation for $S_{lm}(\theta; s\epsilon)$, which is obtained from the substitution of (10) into (9), is given by [41–46]

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dS_{lm}}{d\theta} \right) + \left[K_{lm} + (s\epsilon)^2 \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{lm} = 0. \quad (13)$$

This angular equation is supplemented by the requirement that the angular functions $S_{lm}(\theta; s\epsilon)$ [47] be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions single out the discrete set of angular eigenvalues $\{K_{lm}(s\epsilon)\}$ with $l \geq |m|$ [45]. We shall henceforth consider equatorial scalar modes in the eikonal regime

$$l = m \gg 1 \quad \text{and} \quad s\epsilon \gg 1, \quad (14)$$

in which case the angular eigenvalues are given by [48,49]

$$K_{mm}(s\epsilon) = m^2 - (s\epsilon)^2 + O(m). \quad (15)$$

The radial equation for R_{lm} , which is obtained from the substitution of (10) into (9), is given by [41,42]

$$\Delta \frac{d}{dr} \left(\Delta \frac{dR_{lm}}{dr} \right) + \left[[(r^2 + a^2)\omega - ma]^2 + \Delta[2ma\omega - \mu^2(r^2 + a^2) - K_{lm}] \right] R_{lm} = 0. \quad (16)$$

Note that the radial equation (16) for R_{lm} is coupled to the angular equation (13) for S_{lm} through the angular eigenvalues $\{K_{lm}(s\epsilon)\}$ [50].

4. Stationary bound-state resonances of the composed black-hole–scalar-field system

In the present paper we shall explore the physical properties of the linearized *stationary* scalar configurations which characterize the composed Kerr–Newman–scalar-field system. These stationary bound-state resonances of the bosonic field are characterized by the critical frequency

$$\omega_{\text{field}} = \omega_c \equiv m\Omega_H \quad (17)$$

for superradiant scattering in the black-hole spacetime [see Eq. (3)].

The bound-state solutions of the radial equation (16) are characterized by a decaying field at spatial infinity [36]:

$$R(r \rightarrow \infty) \sim \frac{1}{r} e^{-\epsilon r/r_H} \quad (18)$$

with $\epsilon^2 > 0$ [51]. Regular (finite energy) field configurations are also bounded at the black-hole horizon:

$$R(r = r_H) < \infty. \quad (19)$$

The boundary conditions (18) and (19) single out the discrete family of radial eigenfunctions [along with the associated eigen field-masses, see Eq. (31) below] which characterize the bound-state stationary scalar configurations.

We shall first obtain a simple analytic formula for the discrete spectrum of field masses, $\{\mu(m, s; n)\}$ [52], which characterize the stationary bound-state resonances of the massive scalar fields in the extremal Kerr–Newman black-hole spacetime. To that end, it proves useful to define a new dimensionless radial coordinate [41,42]

$$x \equiv \frac{r - M}{M}, \quad (20)$$

in terms of which the radial equation (16) becomes

$$x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + V R = 0, \quad (21)$$

where $V \equiv [M\omega_c(x+2)]^2 - K + 2Mms\omega_c - (M\mu)^2[(x+1)^2 + s^2]$. Remarkably, this radial equation for $R(x)$ can be solved *analytically* [27,45]:

$$R(x) = C_1 \times x^{-\frac{1}{2}+\beta} e^{-\epsilon x} M\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, 2\epsilon x\right) + C_2 \times (\beta \rightarrow -\beta), \quad (22)$$

where $M(a, b, z)$ is the confluent hypergeometric function [45] and $\{C_1, C_2\}$ are normalization constants. Here

$$\kappa \equiv \frac{\alpha}{\epsilon} - \epsilon \quad \text{with } \alpha \equiv (M\omega_c)^2 = \frac{(ms)^2}{(1+s^2)^2}, \quad (23)$$

and [53]

$$\beta^2 \equiv K + \frac{1}{4} - 2Mms\omega_c - (2M\omega_c)^2 + (M\mu)^2(1+s^2). \quad (24)$$

The notation $(\beta \rightarrow -\beta)$ in (22) means “replace β by $-\beta$ in the preceding term”. Taking cognizance of Eqs. (8), (15) and (17), one can express β in the eikonal regime (14) in the form

$$\beta = \sqrt{\beta_0^2 + \epsilon^2} \quad \text{with } \beta_0^2 \equiv m^2 \frac{1-3s^2}{(1+s^2)^2} [1 + O(m^{-1})]. \quad (25)$$

We shall now analyze the spatial behavior of the radial wave function (22) in the asymptotic regimes $x \rightarrow 0$ and $x \rightarrow \infty$:

(1) The behavior of the radial function (22) in the near-horizon $x \ll 1$ region is given by [45]

$$R(x \rightarrow 0) \rightarrow C_1 \times x^{-\frac{1}{2}+\beta} + C_2 \times x^{-\frac{1}{2}-\beta}. \quad (26)$$

From Eq. (26) one learns that a well-behaved [see Eq. (19)] stationary field configuration is characterized by [54,55]

$$C_2 = 0 \quad \text{and} \quad \Re \beta \geq \frac{1}{2}. \quad (27)$$

(2) The behavior of the radial function (22) in the asymptotic $x \rightarrow \infty$ region is given by [45]

$$R(x \rightarrow \infty) \rightarrow C_1 \times (2\epsilon)^{\kappa-\frac{1}{2}-\beta} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta+\kappa)} x^{-1+\kappa} (-1)^{-\frac{1}{2}-\beta+\kappa} e^{-\epsilon x} + C_1 \times (2\epsilon)^{-\kappa-\frac{1}{2}-\beta} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta-\kappa)} x^{-1-\kappa} e^{\epsilon x}. \quad (28)$$

The bound-state (finite-energy) scalar configurations are characterized by asymptotically decaying eigenfunctions at large distances from the central black hole [see Eq. (18)]. Thus, the coefficient of the growing exponent $e^{\epsilon x}$ in (28) must be identically zero. This boundary condition yields the resonance condition [56]

$$\frac{1}{2} + \beta - \kappa = -n \quad \text{with } n = 0, 1, 2, \dots \quad (29)$$

for the linearized stationary bound-state resonances of the massive scalar fields in the rotating Kerr–Newman black-hole spacetime.

Taking cognizance of Eqs. (23) and (25), one can express the resonance condition (29) in the form $\sqrt{\beta_0^2 + \epsilon^2} = \alpha\epsilon^{-1} - \epsilon - (n + 1/2)$, which in the eikonal regime ($m \gg n + 1/2$) yields the simple relation [see Eqs. (23) and (25)]

$$\epsilon = m \frac{s^2}{(1+s^2)\sqrt{1-s^2}} [1 + O(m^{-1})] \quad (30)$$

for the bound-state resonances in the regime $0 < s < \frac{1}{\sqrt{2}}$. Finally, taking cognizance of the relation (12), one finds

$$M\mu(m, s) = m \frac{s}{(1+s^2)\sqrt{1-s^2}} [1 + O(m^{-1})] \quad (31)$$

for the scalar field-masses which characterize the stationary bound-state resonances of the composed Kerr–Newman–scalar-field system.

5. Effective lengths of the stationary bound-state scalar configurations

Motivated by the intriguing ‘no short hair’ theorem (1) [25], we shall now analyze the effective lengths of the linearized stationary bound-state scalar configurations. Taking cognizance of Eqs. (27) and (29), one can write the radial function (22) for the stationary bound-state configurations in the compact form $R(x) = Ax^{-\frac{1}{2}+\beta} e^{-\epsilon x} L_n^{(2\beta)}(2\epsilon x)$, where A is a normalization constant and $L_n^{(2\beta)}(x)$ are the generalized Laguerre Polynomials [57]. In particular, the fundamental ($n = 0$) bound-state resonance is characterized by the remarkably simple radial eigenfunction [58]

$$R^{(0)}(x) = Ax^{-\frac{1}{2}+\beta} e^{-\epsilon x}. \quad (32)$$

The radial distribution (32) peaks at $x_{\text{peak}} = (\beta - 1/2)/\epsilon$, which implies [see Eqs. (25) and (30)]

$$\chi_{\text{peak}} = \frac{1 - 2s^2}{s^2} [1 + O(m^{-1})]. \quad (33)$$

Eq. (33) reveals the remarkable fact that, the bound-state stationary scalar configurations can be made arbitrarily compact. In particular, one finds

$$\chi_{\text{peak}} \rightarrow 0 \quad \text{for } s \rightarrow \frac{1}{\sqrt{2}}. \quad (34)$$

One therefore concludes that rotating black holes can support extremely short-range stationary scalar configurations (linearized scalar ‘clouds’) in their exterior regions. Our analysis thus provides evidence for the failure of the ‘no short hair’ theorem [25] beyond the regime of spherically-symmetric static black holes.

6. Summary

In a very intriguing letter [25] a remarkable observation was made according to which static spherically-symmetric black holes cannot have short hair. In particular, it was proved [25] that if a spherically-symmetric static black hole has hair, then this hair must extend beyond $3/2$ the horizon radius [see Eq. (1)]. The main goal of the present paper was to test the general validity of this ‘no short hair’ conjecture.

To that end, we have analyzed the physical properties of non-spherically-symmetric rotating black holes coupled to stationary (rather than static) linear matter configurations. In particular, we have shown that rotating Kerr–Newman black holes can support extremely short-range stationary scalar configurations (linearized scalar ‘bristles’) in their exterior regions. Our analysis thus provides compelling evidence for the failure of the ‘no short hair’ conjecture (1) [25] beyond the regime of spherically-symmetric static black holes.

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- [50] We shall henceforth omit the harmonic indexes l and m for brevity.
- [51] One may choose $\epsilon > 0$ without loss of generality.
- [52] Here $n = 0, 1, 2, \dots$ is the resonance parameter.
- [53] One may choose $\Re\beta > 0$ without loss of generality.
- [54] The Boyer–Lindquist coordinates (t, r, θ, ϕ) we are using are not regular at the black-hole horizon $r = r_H$ (in particular, it takes an infinite coordinate time t for ingoing geodesics to cross the black-hole horizon [55]). Hence, in order to verify the regularity of our solution (22) at the black-hole horizon, it is of physical interest to express this solution in coordinates that are regular at the horizon. It is well known that the ingoing Kerr coordinates $(\tilde{t}, r, \theta, \tilde{\phi})$ are regular at the black-hole horizon (in particular, ingoing geodesics pass through the black-hole horizon in a finite coordinate time \tilde{t} [55]). The transformations from the Boyer–Lindquist coordinates t and ϕ to the regular ingoing Kerr coordinates \tilde{t} and $\tilde{\phi}$ are given by [55] $\tilde{t} = t + \alpha(r)$ and $\tilde{\phi} = \phi + \beta(r)$, where $\alpha(r) = -r + \int \frac{r^2 + a^2}{(r - r_H)^2} dr$ and $\beta(r) = \int \frac{a}{(r - r_H)^2} dr$. In terms of the regular ingoing Kerr coordinates, the scalar field can be decomposed in the form [55] $\Psi(\tilde{t}, r, \theta, \tilde{\phi}) = \int \sum_{l,m} e^{im\tilde{\phi}} S_{lm}(\theta; s\epsilon) \tilde{R}_{lm}(r; s, \mu, \omega) e^{-i\omega\tilde{t}} d\omega$, which yields the relation [see Eq. (10)] $\tilde{R}_{lm}(r) = e^{i[\omega\alpha(r) - m\beta(r)]} R_{lm}(r)$. Here $\tilde{R}_{lm}(r)$ is the radial wave function in the regular ingoing Kerr coordinates. In particular, for $\omega = m\Omega_H = m \frac{a}{r_H^2 + a^2}$ [see Eqs. (3) and (8)] one finds $\omega\alpha(r \simeq r_H) - m\beta(r \simeq r_H) = -m\Omega_H r_H$ in the near-horizon region $r \rightarrow r_H$, which implies $\tilde{R}_{lm}(r \simeq r_H) = e^{-im\Omega_H r_H} R_{lm}(r \simeq r_H)$. From this transformation one finds that the radial solution $\tilde{R}(x \rightarrow 0) = \tilde{C}_1 \times x^{-\frac{1}{2} + \beta}$ for the scalar field in the near-horizon $r \rightarrow r_H$ ($x \rightarrow 0$) region [see Eqs. (22) and (27)] is a smooth function of the regular ingoing Kerr coordinates.
- [55] S.R. Dolan, Phys. Rev. D 76 (2007) 084001.
- [56] Here we have used Eq. 6.1.7 of [45].
- [57] See Eq. 13.6.9 of [45].
- [58] Here we have used the identity $L_0^{(2\beta)}(x) \equiv 1$.